National Research Journal of Information Technology & Information Science ISSN No: 2350-1278
Volume No: 12, Issue No: 1, Year: 2025 (January-June) Peer Reviewed & Refereed Journal (IF: 7.9)
PP: 97-108 Journal Website www.nrjitis.in

CERTAIN FRACTIONAL INTEGRAL FORMULAS INVOLVING KAMPE DE FE´RIET FUNCTION

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Abstract. Fractional integral formulas are established by using fractional integral operators involving Kampe de Feriet Functions. Some special cases are also discussed in our present study.

1. Introduction and Preliminaries

In recent years, fractional calculus has been applied to almost all fields of mathematics, applied science and engineering. Many researchers have extensively studied the properties, applications and extensions of various fractional integral and differential operators involving the various special functions, for example McBride [11], Kalla [1, 2], Kalla and Saxena [3, 4], Saigo [16, 17, 18], Saigo and Maeda [19], Kiryakova [7, 8], Miller and Ross [12] etc.

For our present study, we recall the generalized hypergeometric fractional integrals, introduced by Marichev [10], including the Saigo operators [16, 17, 18] and which was later on extended and studied by Saigo and Maeda [19].

Marchichev-Saigo-Maeda fractional integral operators involving the Appell's function $F_3(.)$ in the kernal are defined as:

Let $\alpha,\alpha',\beta,\beta',\gamma \in \mathrm{C}$ and $x>0$, then for $\mathrm{R}(\gamma)>0$

$$\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)$$
$$=\frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x(x-t)^{\gamma-1}t^{-\alpha'}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)\,dt \tag{1.1}$$

and

$$\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)$$
$$=\frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^\infty(t-x)^{\gamma-1}t^{-\alpha}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)\,dt \tag{1.2}$$

where the function $f(t)$ is so constrained that the integrals in (1.1) and (1.2) exist.

And $F_3(.)$ denotes Appell's hypergeometric function [22] in two variables defined as:

$$F_3(\alpha,\alpha',\beta,\beta';\gamma;x;y)$$
$$=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta)_n}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{x^n}{n!}\quad (\max\{|x|,|y|\}<1) \tag{1.3}$$

The above fractional integral operators in (1.1) and (1.2) can be written as follows:

$$\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(\frac{d}{dx}\right)^k\left(I_{0,x}^{\alpha,\alpha',\beta+k,\beta',\gamma+k}f\right)(x)$$
$$(\Re(\gamma)>0;k=[\Re(\gamma)]+1) \tag{1.4}$$

and

$$\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(-\frac{d}{dx}\right)^{k}\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta'+k,\gamma+k}f\right)(x)$$
$$(\Re(\gamma)>0;k=[\Re(\gamma)]+1) \tag{1.5}$$

The Appell function defined in equation (1.3) reduces to the Gauss hypergeometric function ${}_2F_1$ as given in following relations.

$$F_3(\alpha,\gamma-\alpha,\beta,\gamma-\beta;\gamma;x;y)={}_2F_1(\alpha,\beta;\gamma;x+y-xy) \tag{1.6}$$

Also

$$F_3(\alpha,0,\beta,\beta',\gamma;x,y)={}_2F_1(\alpha,\beta;\gamma;x) \tag{1.7}$$

And

$$F_3(0,\alpha',\beta,\beta',\gamma;x,y)={}_2F_1(\alpha',\beta';\gamma;y) \tag{1.8}$$

In view of the above reduction formula as given in equation (1.7), the general operators reduce to the Saigo operators [15] defined as follows:

For $x>0,\ \alpha,\beta,\gamma\in\mathbb{C}$

$$\left(I_{0,x}^{\alpha,\beta,\gamma}f\right)(x)=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;1-\frac{t}{x}\right)f(t)\,dt$$
$$(\Re(\alpha)>0) \tag{1.9}$$

and

$$\left(I_{x,\infty}^{\alpha,\beta,\gamma}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^\infty(t-x)^{\alpha-1}t^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;1-\frac{x}{t}\right)f(t)\,dt$$
$$(\Re(\alpha)>0) \tag{1.10}$$

where the ${}_2F_1(.)$, a special case of the generalized hypergeomteric function, is the Gauss hypergeometric function and function $f(t)$ is so constrained that the integrals in (1.9) and (1.10) converge.

The Saigo fractional integral operators, given in (1.9) and (1.10) can also be written as:

For $x>0,\ \alpha,\beta,\gamma\in\mathbb{C}$

$$\left(I_{0,x}^{\alpha,\beta,\gamma}f\right)(x)=\left(\frac{d}{dx}\right)^{k}\left(I_{0,x}^{\alpha+k,\beta-k,\gamma-k}f\right)(x)$$
$$(\Re(\alpha)\le 0;\ k=[\Re(-\alpha)]+1) \tag{1.11}$$

and

$$\left(I_{x,\infty}^{\alpha,\beta,\gamma}f\right)(x)=\left(-\frac{d}{dx}\right)^{k}\left(I_{x,\infty}^{\alpha-k,\beta-k,\gamma}f\right)(x)$$
$$(\Re(\alpha)\le 0;\ k=[\Re(-\alpha)]+1) \tag{1.12}$$

The Erd´elyi-Kober type fractional integral operators are defined as follows [9] For

$x>0,\ \alpha,\gamma\in\mathbb{C}$

$$\left(E_{0,x}^{\alpha,\gamma}f\right)(x)=\frac{x^{-\alpha-\gamma}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}t^{\gamma}f(t)\,dt \qquad (\Re(\alpha)>0) \tag{1.13}$$

$$\left(K_{x,\infty}^{\alpha,\gamma}f\right)(x)=\frac{x^{\gamma}}{\Gamma(\alpha)}\int_x^\infty(t-x)^{\alpha-1}t^{-\alpha-\gamma}f(t)\,dt \tag{1.14}$$

provided that integrals in (1.13) and (1.14) converge.

The Riemann - Liouville fractional integral operator and the Weyl fractional integral operator [13] are defined as follows:

$x > 0,\ \alpha \in \mathbb{C}$

$$\left(R_{0,x}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt \qquad (\Re(\alpha) > 0) \tag{1.15}$$

$$\left(W_{x,\infty}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t)\, dt \tag{1.16}$$

provided both integrals converge.

In view of the reduction formula (1.7), the Marichev-Saigo-Maeda and the Saigo fractional integral operators are related as [20]:

$$\left(I_{0,x}^{\alpha,0,\beta,\beta',\gamma} f\right)(x) = \left(I_{0,x}^{\gamma,\alpha-\gamma,-\beta} f\right)(x \qquad (\gamma \in \mathrm{C}) \tag{1.17}$$

and

$$\left(I_{x,\infty}^{\alpha,0,\beta,\beta',\gamma} f\right)(x) = \left(I_{x,\infty}^{\gamma,\alpha-\gamma,-\beta} f\right)(x \qquad (\gamma \in \mathrm{C}) \tag{1.18}$$

The relation between Saigo fractional integral operators and Erd´elyi-Kober fractional integral operators are given by Kilbas et. at.[6] as:

$$\left(E_{0,x}^{\alpha,\gamma} f\right)(x) = \left(I_{0,x}^{\alpha,0,\gamma} f\right)(x) \tag{1.19}$$

$$(K_{x,\alpha,\gamma\infty} f)(x) = (I_{x,\alpha,\infty\, 0,\, \gamma} f)(x) \tag{1.20}$$

The operator $I_{0,x}^{\alpha,\beta,\gamma}(\cdot$ contains the Riemann-Liouville $R_{0,x}^{\alpha}(\cdot$ and $I_{x,}{}^{\alpha,\beta,\gamma_\infty}(.)$ contains the

Weyl $W_{0,x}^{\alpha}(\cdot$ fractional integral operators by means of the following relationships:

$$\left(R_{0,x}^{\alpha} f\right)(x) = \left(I_{0,x}^{\alpha,-\alpha,\gamma} f\right)(x), \tag{1.21}$$

$$\left(W_{x,\infty}^{\alpha} f\right)(x) = \left(I_{x,\infty}^{\alpha,-\alpha,\gamma} f\right)(x) \tag{1.22}$$

Power functions formulas of the above discussed fractional integral operators are required for our present study as given in the following lemmas [19, 16, 20].

Lemma 1. *Let α,α′,β,β′,γ and ρ ∈ C be such that* $\Re(\gamma) > 0$, *then the following formulas hold true:*

$$\begin{aligned}&\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right)(x)\\ &= \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\beta')\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)} x^{\rho+\gamma-\alpha-\alpha'-1}\\ &\quad (\Re(\rho) > \max\{0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')\})\end{aligned} \tag{1.23}$$

and

$$\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\right)(x)$$
$$= \frac{\Gamma(1-\rho-\beta)\Gamma(1-\rho-\gamma+\alpha+\alpha')\Gamma(1-\rho-\gamma+\alpha+\beta')}{\Gamma(1-\rho)\Gamma(1-\rho-\gamma+\alpha+\alpha'+\beta')\Gamma(1-\rho+\alpha-\beta)}x^{\rho+\gamma-\alpha-\alpha'-1}$$
$$(\Re(\rho) < 1+\min\{\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\}) \tag{1.24}$$

Lemma 2. *Let $\alpha,\beta,\gamma,\rho \in$ C be such that $\Re(\alpha) > 0$, then the following formulas hold true:*

$$\left(I_{0,x}^{\alpha,\beta,\gamma}t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\beta)}{\Gamma(\rho-\beta)\Gamma(\rho+\gamma+\alpha)}x^{\rho-\beta-1}$$
$$(\Re(\rho) > \max\{0, \Re(\beta-\gamma)\}) \tag{1.25}$$

and

$$\left(I_{x,\infty}^{\alpha,\beta,\gamma}t^{\rho-1}\right)(x) = \frac{\Gamma(1-\rho+\beta)\Gamma(1-\rho+\gamma)}{\Gamma(1-\rho)\Gamma(1-\rho+\gamma+\alpha+\beta)}x^{\rho-\beta-1}$$
$$(\Re(\rho) < 1+\min\{\Re(\beta), \Re(\gamma)\}) \tag{1.26}$$

For our present study, we begin by recalling generalized Kamp´e de F´eriet function [22, 21] is defined as:

$$F_{l;m;n}^{p;q;k}\left[\begin{array}{l} (a_p);\ (b_q);\ (c_k); \\ (\alpha_l);\ (\beta_m);\ (\gamma_n); \end{array} x, y\right] \tag{1.27}$$
$$= \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}$$
$$p,q,k,l,m,n \in \mathbb{N}_0.$$

the above function converges under the following conditions:
$p+q < l+m+1, p+k < l+n+1, |x| < \infty, |y| < \infty$

also we recall the generalized hypergeometric series ${}_pF_q$($p,q \in$ N0) is defined as [14, 23]

$${}_pF_q\left[\begin{array}{l} \alpha_1,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_q; \end{array} z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}$$
$$= {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z) \tag{1.28}$$

and $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in$ C) by (see [23]):

$$(\lambda)_n := \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases} \tag{1.29}$$
$$= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C}\setminus\mathbb{Z}_0^-), \tag{1.30}$$

and Z^-_0 denotes the set of Non-positive integers. where $\Gamma(\lambda)$ is familiar Gamma function.

2. Main Results

In this section we present certain fractional integral formulas involving the generalized Kamp´e de F´eriet function by using fractional integral operators.

Theorem 1. *Let* $x > 0, \sigma, \sigma', \nu, \nu', \eta, \rho \in \mathbb{C}$, $p,q,k,l,m,n \in \mathbb{N}_0$ *and*

$$p + q < l + m + 1, p + k < l + n + 1, |ty| < \infty, |tz| < \infty$$

or

$$p + q = l + m + 1, \; p + k = l + n + 1$$

be such that $\Re(\eta) > 0$ *and* $\Re(\rho+r+s) > \max\{0, \Re(\sigma+\sigma'+\nu-\eta), \Re(\sigma'-\nu')\}$ *then the following fractional integral formula holds true:*

$$\left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta} t^{\rho-1} F_{l;m;n}^{p;q;k}\left[\begin{array}{c}(a_p);(b_q);(c_k);\\(\alpha_l);(\beta_m);(\gamma_n);\end{array} ty, tz\right]\right)(x)$$
$$= x^{\rho+\eta-\sigma-\sigma'-1}\frac{\Gamma(\rho)\Gamma(\rho+\eta-\sigma-\sigma'-\nu)\Gamma(\rho+\nu'-\sigma')}{\Gamma(\rho+\nu')\Gamma(\rho+\eta-\sigma-\sigma')\Gamma(\rho+\eta-\sigma'-\nu)}$$
$$\times F_{l+3;m;n}^{p+3;q;k}\left[\begin{array}{c}(a_p);(\rho);(\rho+\eta-\sigma-\sigma'-\nu);(\rho+\nu'-\sigma');(b_q);(c_k);\\(\alpha_l);(\rho+\nu');(\rho+\eta-\sigma-\sigma');(\rho+\eta-\sigma'-\nu);(\beta_m);(\gamma_n);\end{array} xy, xz\right] \tag{2.1}$$

Proof. For convenience, we denote the left-hand side of the result (2.1) by *I*. Using (1.27) and then changing the order of integration and summation, we have

$$I = \sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{y^r}{r!}\frac{z^s}{s!}$$
$$\times\left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta} t^{\rho+r+s-1}\right)(x) \tag{2.2}$$

applying the result (1.23), the above equation (2.2) reduces to

$$I = \sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{y^r}{r!}\frac{z^s}{s!}$$
$$\times\frac{\Gamma(\rho+r+s)\Gamma(\rho+r+s+\eta-\sigma-\sigma'-\nu)\Gamma(\rho+r+s+\nu'-\sigma')}{\Gamma(\rho+r+s+\nu')\Gamma(\rho+r+s+\eta-\sigma-\sigma')\Gamma(\rho+r+s+\eta-\sigma'-\nu)}$$
$$\times x^{\rho+r+s+\eta-\sigma-\sigma'-1} \tag{2.3}$$

after simplification, the above equation (2.3) reduces to

$$
\begin{aligned}
\Gamma &= x^{\rho+\eta-\sigma-\sigma'-1} \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s} \prod_{j=1}^{q}(b_j)_r \prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s} \prod_{j=1}^{m}(\beta_j)_r \prod_{j=1}^{n}(\gamma_j)_s} \\
&\times \frac{\Gamma(\rho)\Gamma(\rho+\eta-\sigma-\sigma'-\nu)\Gamma(\rho+\nu'-\sigma')}{\Gamma(\rho+\nu')\Gamma(\rho+\eta-\sigma-\sigma')\Gamma(\rho+\eta-\sigma'-\nu)} \\
&\times \frac{(\rho)_{r+s}(\rho+\eta-\sigma-\sigma'-\nu)_{r+s}(\rho+\nu'-\sigma')_{r+s}}{(\rho+\nu')_{r+s}(\rho+\eta-\sigma-\sigma')_{r+s}(\rho+\eta-\sigma'-\nu)_{r+s}} \frac{(xy)^r}{r!} \frac{(xz)^s}{s!} \qquad (2.4) \\
&= x^{\rho+\eta-\sigma-\sigma'-1} \frac{\Gamma(\rho)\Gamma(\rho+\eta-\sigma-\sigma'-\nu)\Gamma(\rho+\nu'-\sigma')}{\Gamma(\rho+\nu')\Gamma(\rho+\eta-\sigma-\sigma')\Gamma(\rho+\eta-\sigma'-\nu)} \\
&\times \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s} \prod_{j=1}^{q}(b_j)_r \prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s} \prod_{j=1}^{m}(\beta_j)_r \prod_{j=1}^{n}(\gamma_j)_s} \\
&\times \frac{(\rho)_{r+s}(\rho+\eta-\sigma-\sigma'-\nu)_{r+s}(\rho+\nu'-\sigma')_{r+s}}{(\rho+\nu')_{r+s}(\rho+\eta-\sigma-\sigma')_{r+s}(\rho+\eta-\sigma'-\nu)_{r+s}} \frac{(xy)^r}{r!} \frac{(xz)^s}{s!} \qquad (2.5)
\end{aligned}
$$

interpreting the above equation, in the view of of (1.27), we have the required result.

Theorem 2. *Let* $x > 0, \sigma, \sigma', \nu, \nu', \eta, \rho \in \mathbb{C}$, $p, q, k, l, m, n \in \mathbb{N}_0$ *and*

$$p + q < l + m + 1, p + k < l + n + 1, |y/t| < \infty, |z/t| < \infty$$

or

$$p + q = l + m + 1,\ p + k = l + n + 1$$

$$\{ |y/t|^{1/(p-l)} + |z/t| \}^{1/(q-l)} < 1, \qquad p > l$$

and

$$\max\{y/t, z/t\} < 1, \qquad p \le l$$

be such that $\Re(\eta) > 0$ *and* $\Re(\rho - r - s) < 1 + min\{\Re(-\nu), \Re(\sigma + \sigma' - \eta), \Re(\sigma + \nu' - \eta)\}$ *then the following fractional integral formula holds true:*

$$
\begin{aligned}
&\left(I_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta} t^{\rho-1} F_{l;m;n}^{p;q;k} \left[\begin{matrix} (a_p); (b_q); (c_k); \\ (\alpha_l); (\beta_m); (\gamma_n); \end{matrix} \ \frac{y}{t}, \frac{z}{t} \right] \right)(x) \\
&= x^{\rho+\eta-\sigma-\sigma'-1} \frac{\Gamma(1-\rho-\nu)\Gamma(1-\rho-\eta+\sigma+\sigma')\Gamma(1-\rho-\eta+\sigma+\nu')}{\Gamma(1-\rho)\Gamma(1-\rho-\eta+\sigma+\sigma'+\nu')\Gamma(1-\rho+\sigma-\nu)} \\
&\times F_{l+3;m;n}^{p+3;q;k} \left[\begin{matrix} (a_p); (1-\rho-\nu); (1-\rho-\eta+\sigma+\sigma'); (1-\rho-\eta+\sigma+\nu'); (b_q); (c_k); \\ (\alpha_l); (1-\rho); (1-\rho-\eta+\sigma+\sigma'+\nu'); (1-\rho+\sigma-\nu); (\beta_m); (\gamma_n); \end{matrix} \ \frac{y}{x}, \frac{z}{x} \right] \qquad (2.6)
\end{aligned}
$$

Proof. Proof is similar to that of Theorem 1.

2.1. **Special Cases.** If we put $\sigma = \sigma + \nu, \sigma' = \nu' = 0, \nu = -\eta, \eta = \alpha$ in Theorems 1 and 2 and employing the relations (2.1) and (2.6) yield certain interesting results concerning the Saigo fractional integral operator given in the following corollaries.

Corollary 1. *Let* $x > 0, \sigma, \nu, \eta, \rho \in \mathbb{C}$, $p, q, k, l, m, n \in \mathbb{N}_0$ *and*

$$p + q < l + m + 1, p + k < l + n + 1, |ty| < \infty, |tz| < \infty$$

or

$$p + q = l + m + 1,\ p + k = l + n + 1$$

$$\{\ |ty|^{1/(|p-l|)} + |tz|^{1/(q-l)} < 1,\ p > l$$

and

$$\max\{|ty|, |tz|\} < 1, \qquad p \le l$$

be such that $\Re(\sigma) > 0$ *and* $\Re(\rho + r + s) > max\{0, \Re(\nu - \eta)\}$ *then the following fractional integral formula holds true:*

$$\left(I_{0,x}^{\sigma,\nu,\eta} t^{\rho-1} F_{l;m;n}^{p;q;k}\left[\begin{array}{l}(a_p);(b_q);(c_k);\\(\alpha_l);(\beta_m);(\gamma_n);\end{array} ty, tz\right]\right)(x)$$
$$= x^{\rho-\nu-1}\frac{\Gamma(\rho)\Gamma(\rho+\eta-\nu)}{\Gamma(\rho-\nu)\Gamma(\rho+\eta+\sigma)}$$
$$\times F_{l+2;m;n}^{p+2;q;k}\left[\begin{array}{c}(a_p);(\rho);(\rho+\eta-\nu);(b_q);(c_k);\\(\alpha_l);(\rho-\nu);(\rho+\eta+\sigma);(\beta_m);(\gamma_n);\end{array} xy, xz\right] \tag{2.7}$$

Corollary 2. *Let* $x > 0, \sigma, \nu, \eta, \rho \in \mathbb{C}$, $p,q,k,l,m,n \in \mathbb{N}_0$ *and*

$$p + q < l + m + 1, p + k < l + n + 1, |y/t| < \infty, |z/t| < \infty$$

or

$$p + q = l + m + 1,\ p + k = l + n + 1$$

$$\{\ |y/t|^{1/(|p-l|)} + |z/t|\}^{1/(q-l)} < 1, \qquad p > l$$

and

$$\max\{|y/t|, |z/t|\} < 1, \qquad p \le l$$

be such that $\Re(\sigma) > 0$ *and* $\Re(\rho - r - s) < 1 + min\{\Re(\nu), \Re(\eta)\}$ *then the following fractional integral formula holds true:*

$$\left(I_{x,\infty}^{\sigma,\nu,\eta} t^{\rho-1} F_{l;m;n}^{p;q;k}\left[\begin{array}{l}(a_p);(b_q);(c_k);\\(\alpha_l);(\beta_m);(\gamma_n);\end{array} \frac{y}{t}, \frac{z}{t}\right]\right)(x)$$
$$= x^{\rho-\nu-1}\frac{\Gamma(1-\rho+\nu)\Gamma(1-\rho+\eta)}{\Gamma(1-\rho)\Gamma(1-\rho+\sigma+\nu+\eta)}$$
$$\times F_{l+2;m;n}^{p+2;q;k}\left[\begin{array}{c}(a_p);(1-\rho+\nu);(1-\rho+\eta);(b_q);(c_k);\\(\alpha_l);(1-\rho);(1-\rho+\sigma+\nu+\eta);(\beta_m);(\gamma_n);\end{array} \frac{y}{x}, \frac{z}{x}\right] \tag{2.8}$$

Further, if we put $\nu = 0$ in (2.7) and (2.8) then these Saigo fractional integrals reduce to the following Erd´elyi-Kober type fractional integral operators as given below:

Corollary 3. *Let* $x > 0, \sigma, \eta, \rho \in \mathbb{C}$, $p,q,k,l,m,n \in \mathbb{N}_0$ *and*

$$p + q < l + m + 1, p + k < l + n + 1, |ty| < \infty, |tz| < \infty$$

or

$$p + q = l + m + 1,\ p + k = l + n + 1$$

$$\{\ |ty|^{1/(|p-l|)} + |tz|^{1/(q-l)} < 1,\ p > l$$

and

max $\{|ty|, |tz|\} < 1, \quad p \le l$

be such that $\mathrm{R}(\sigma) > 0$ *and* $\mathrm{R}(\rho + r + s + \eta) > 0$ *then the following fractional integral formula holds true:*

$$\left(E_{0,x}^{\sigma,\eta} t^{\rho-1} F_{l;m;n}^{p;q;k}\left[\begin{array}{l}(a_p);(b_q);(c_k);\\(\alpha_l);(\beta_m);(\gamma_n);\end{array} ty, tz\right]\right)(x)$$
$$= x^{\rho-1}\frac{\Gamma(\rho+\eta)}{\Gamma(\rho+\sigma+\eta)} \times F_{l+1;m;n}^{p+1;q;k}\left[\begin{array}{l}(a_p);(\rho+\eta);(b_q);(c_k);\\(\alpha_l);(\rho+\sigma+\eta);(\beta_m);(\gamma_n);\end{array} xy, xz\right] \tag{2.9}$$

Corollary 4. *Let* $x > 0, \sigma, \eta, \rho \in \mathbb{C}$, $p,q,k,l,m,n \in \mathbb{N}_0$ *and*

$p + q < l + m + 1, p + k < l + n + 1, |y/t| < \infty, |z/t| < \infty$

or

$p + q = l + m + 1,\ p + k = l + n + 1$

$\{ |y/t|^{1/(p-l)} + |z/t|^{1/(q-l)} \} < 1, \quad p > l$

and

max $\{|y/t|, |z/t|\} < 1, \quad p \le l$

be such that $\mathrm{R}(\sigma) > 0$ *and* $\mathrm{R}(\eta) > \mathrm{R}(\rho - r - s) > -1$ *then the following fractional integral formula holds true:*

$$\left(K_{x,\infty}^{\sigma,\eta} t^{\rho-1} F_{l;m;n}^{p;q;k}\left[\begin{array}{l}(a_p);(b_q);(c_k);\\(\alpha_l);(\beta_m);(\gamma_n);\end{array} \frac{y}{t}, \frac{z}{t}\right]\right)(x)$$
$$= x^{\rho-1}\frac{\Gamma(1-\rho+\eta)}{\Gamma(1-\rho+\sigma+\eta)} \times F_{l+1;m;n}^{p+1;q;k}\left[\begin{array}{l}(a_p);(1-\rho+\eta);(b_q);(c_k);\\(\alpha_l);(1-\rho+\sigma+\eta);(\beta_m);(\gamma_n);\end{array} \frac{y}{x}, \frac{z}{x}\right] \tag{2.10}$$

Further, if we put $\nu = -\sigma$ in (2.7) and (2.8) then these Saigo fractional integrals reduce to the following Riemann-Liouville and the Weyl type fractional integral operators as given in following results:

Corollary 5. *Let* $x > 0, \sigma, \rho \in \mathbb{C}$, $p,q,k,l,m,n \in \mathbb{N}_0$ *and*

$p + q < l + m + 1, p + k < l + n + 1, |ty| < \infty, |tz| < \infty$

be such that $\mathrm{R}(\sigma) > 0$ *and* $\mathrm{R}(\rho + r + s) > 0$ *then the following fractional integral formula holds true:*

$$\left(R_{0,x}^{\sigma} t^{\rho-1} F_{l;m;n}^{p;q;k}\left[\begin{array}{l}(a_p);(b_q);(c_k);\\(\alpha_l);(\beta_m);(\gamma_n);\end{array} ty, tz\right]\right)(x)$$
$$= x^{\rho+\sigma-1}\frac{\Gamma(\rho)}{\Gamma(\rho+\sigma)} \times F_{l+1;m;n}^{p+1;q;k}\left[\begin{array}{l}(a_p);(\rho);(b_q);(c_k);\\(\alpha_l);(\rho+\sigma);(\beta_m);(\gamma_n);\end{array} xy, xz\right] \tag{2.11}$$

Corollary 6. *Let* $x > 0, \sigma, \rho \in \mathbb{C}$, $p,q,k,l,m,n \in \mathbb{N}_0$ *and*

$p + q < l + m + 1, p + k < l + n + 1, |y/t| < \infty, |z/t| < \infty$

or

$p + q = l + m + 1,\ p + k = l + n + 1$

$\{ |y/t|^{1/(p-l)} + |z/t|^{1/(q-l)} \} < 1, \quad p > l$

and

max $\{|y/t|, |z/t|\} < 1, \quad p \le l$

be such that R($\rho-r-s$) >R(σ) >−1 *then the following fractional integral formula holds true:*

$$\left(W_{x,\infty}^{\sigma}t^{\rho-1}F_{l;m;n}^{p;q;k}\left[\begin{array}{c}(a_p);(b_q);(c_k);\\(\alpha_l);(\beta_m);(\gamma_n);\end{array}\frac{y}{t},\frac{z}{t}\right]\right)(x)$$
$$=x^{\rho+\sigma-1}\frac{\Gamma(1-\rho+\sigma)}{\Gamma(1-\rho)}\times F_{l+1;m;n}^{p+1;q;k}\left[\begin{array}{c}(a_p);(1-\rho+\sigma);(b_q);(c_k);\\(\alpha_l);(1-\rho);(\beta_m);(\gamma_n);\end{array}\frac{y}{x},\frac{z}{x}\right] \tag{2.12}$$

If we use the following known result [22]

$$F_{l;0;0}^{p;0;0}\left[\begin{array}{c}a_1,\ldots a_p\\\alpha_1,\ldots,\alpha_l\end{array};y,z\right]={}_pF_l\left[\begin{array}{c}a_1,\ldots a_p\\\alpha_1,\ldots,\alpha_l\end{array};y+z\right] \tag{2.13}$$

then results in (2.1), (2.6), (2.7), (2.8), (2.9), (2.10), (2.11) and (2.12) reduce to the following form:

Corollary 7. *Let* $x>0,\sigma,\sigma',\nu,\nu',\eta,\rho\in\mathbb{C}$, $p,l\in\mathbb{N}_0$ *and*

$$|t(y+z)|<\infty \qquad \text{if } p\le l$$

or

$$|t(y+z)|<1 \qquad \text{if } p=l+1,$$

be such that R(η) >0 *and* R($\rho+r$) >max{0,R($\sigma+\sigma'+\nu-\eta$),R($\sigma'-\nu'$)}, $\alpha_i\neq 0,-1,-2,\ldots;i=1,2,\ldots,l$. *then the following fractional integral formula holds true:*

$$\left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}t^{\rho-1}{}_pF_l\left[\begin{array}{c}a_1,\ldots a_p\\\alpha_1,\ldots,\alpha_l\end{array};t(y+z)\right]\right)(x)$$
$$=x^{\rho+\eta-\sigma-\sigma'-1}\frac{\Gamma(\rho)\Gamma(\rho+\eta-\sigma-\sigma'-\nu)\Gamma(\rho+\nu'-\sigma')}{\Gamma(\rho+\nu')\Gamma(\rho+\eta-\sigma-\sigma')\Gamma(\rho+\eta-\sigma'-\nu)}$$
$$\times{}_{p+3}F_{l+3}\left[\begin{array}{c}a_1,\ldots a_p,\rho,\rho+\eta-\sigma-\sigma'-\nu,\rho+\nu'-\sigma';\\\alpha_1,\ldots,\alpha_l,\rho+\nu',\rho+\eta-\sigma-\sigma',\rho+\eta-\sigma'-\nu;\end{array}x(y+z)\right] \tag{2.14}$$

Corollary 8. *Let* $x>0,\sigma,\sigma',\nu,\nu',\eta,\rho\in\mathbb{C}$, $p,l\in\mathbb{N}_0$ *and*

$$|(y+z)/t|<\infty \qquad \text{if } p\le l$$

or

$$|(y+z)/t|<1 \qquad \text{if } p=l+1,$$

be such that R(η) >0 *and* R($\rho-r$) <1 + min{$\Re(-\nu)$,R($\sigma+\sigma'-\eta$),R($\sigma+\nu'-\eta$)}, $\alpha_i\neq 0,-1,-2,\ldots;i=1,2,\ldots,l$. *then the following fractional integral formula holds true:*

$$\left(I_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}t^{\rho-1}{}_pF_l\left[\begin{array}{c}a_1,\ldots a_p\\\alpha_1,\ldots,\alpha_l\end{array};\frac{y+z}{t}\right]\right)(x)$$
$$=x^{\rho+\eta-\sigma-\sigma'-1}\frac{\Gamma(1-\rho-\nu)\Gamma(1-\rho-\eta+\sigma+\sigma')\Gamma(1-\rho-\eta+\sigma+\nu')}{\Gamma(1-\rho)\Gamma(1-\rho-\eta+\sigma+\sigma'+\nu')\Gamma(1-\rho+\sigma-\nu)}$$
$$\times{}_{p+3}F_{l+3}\left[\begin{array}{c}a_1,\ldots a_p,\rho,\rho+\eta-\sigma-\sigma'-\nu,\rho+\nu'-\sigma';\\\alpha_1,\ldots,\alpha_l,\rho+\nu',\rho+\eta-\sigma-\sigma',\rho+\eta-\sigma'-\nu;\end{array}\frac{y+z}{x}\right] \tag{2.15}$$

For $\sigma=\sigma+\nu,\sigma'=\nu'=0,\nu=-\eta,\eta=\alpha$ (2.14) and (2.15) yield certain interesting results concerning the Saigo fractional integral operator given in the following corollaries.

Corollary 9. *Let* $x > 0, \sigma, \nu, \eta, \rho \in \mathbb{C}$, $p, l \in \mathbb{N}_0$ *and*

$$|t(y+z)| < \infty \qquad \text{if } p \le l$$

or

$$|t(y+z)| < 1 \qquad \text{if } p = l+1,$$

be such that $\Re(\sigma) > 0$ *and* $\Re(\rho + r) > \max\{0, \Re(\nu - \eta)\}$, $\alpha_i \neq 0, -1, -2, \ldots; i = 1, 2, \ldots, l$. *then the following fractional integral formula holds true:*

$$\left(I_{0,x}^{\sigma,\nu,\eta} t^{\rho-1} {}_pF_l\left[\begin{array}{c} a_1, \ldots a_p \\ \alpha_1, \ldots, \alpha_l \end{array}; t(y+z)\right]\right)(x)$$
$$= x^{\rho-\nu-1} \frac{\Gamma(\rho)\Gamma(\rho+\eta-\nu)}{\Gamma(\rho-\nu)\Gamma(\rho+\eta+\sigma)}$$
$$\times {}_{p+2}F_{l+2}\left[\begin{array}{c} a_1, \ldots a_p, \rho, \rho+\eta-\nu; \\ \alpha_1, \ldots, \alpha_l, \rho-\nu, \rho+\eta+\sigma; \end{array} x(y+z)\right] \qquad (2.16)$$

Corollary 10. *Let* $x > 0, \sigma, \nu, \eta, \rho \in \mathbb{C}$, $p, l \in \mathbb{N}_0$ *and*

$$|(y+z)/t| < \infty \qquad \text{if } p \le l$$

or

$$|(y+z)/t| < 1 \qquad \text{if } p = l+1,$$

be such that $\Re(\sigma) > 0$ *and* $\Re(\rho - r) < 1 + \min\{\Re(\nu), \Re(\eta)\}$, $\alpha_i \neq 0, -1, -2, \ldots; i = 1, 2, \ldots, l$. *then the following fractional integral formula holds true:*

$$\left(I_{x,\infty}^{\sigma,\nu,\eta} t^{\rho-1} {}_pF_l\left[\begin{array}{c} a_1, \ldots a_p \\ \alpha_1, \ldots, \alpha_l \end{array}; \frac{y+z}{t}\right]\right)(x)$$
$$= x^{\rho-\nu-1} \frac{\Gamma(1-\rho-\nu)\Gamma(1-\rho+\eta)}{\Gamma(1-\rho)\Gamma(1-\rho+\eta+\sigma+\nu)}$$
$$\times {}_{p+2}F_{l+2}\left[\begin{array}{c} a_1, \ldots a_p, 1-\rho-\nu, 1-\rho+\eta; \\ \alpha_1, \ldots, \alpha_l, 1-\rho, 1-\rho+\eta+\sigma+\nu; \end{array} \frac{y+z}{x}\right] \qquad (2.17)$$

For $\nu = 0$ in (2.16) and (2.17) then these Saigo fractional integrals reduce to the following Erd´elyi-Kober type fractional integral operators as given below:

Corollary 11. *Let* $x > 0, \sigma, \eta, \rho \in \mathbb{C}$, $p, l \in \mathbb{N}_0$ *and*

$$|t(y+z)| < \infty \qquad \text{if } p \le l$$

or

$$|t(y+z)| < 1 \qquad \text{if } p = l+1,$$

be such that $\Re(\sigma) > 0$ *and* $\Re(\rho + r + \eta) > 0$, $\alpha_i \neq 0, -1, -2, \ldots; i = 1, 2, \ldots, l$. *then the following fractional integral formula holds true:*

$$\left(E_{0,x}^{\sigma,\eta} t^{\rho-1} {}_pF_l\left[\begin{array}{c} a_1, \ldots a_p \\ \alpha_1, \ldots, \alpha_l \end{array}; t(y+z)\right]\right)(x)$$
$$= x^{\rho-1} \frac{\Gamma(\rho+\eta)}{\Gamma(\rho+\sigma+\eta)} \times {}_{p+2}F_{l+2}\left[\begin{array}{c} a_1, \ldots a_p, \rho+\eta; \\ \alpha_1, \ldots, \alpha_l, \rho+\sigma+\eta; \end{array} x(y+z)\right] \qquad (2.18)$$

Corollary 12. *Let* $x > 0, \sigma, \eta, \rho \in \mathbb{C}$, $p, l \in \mathbb{N}_0$ *and*

$|(y+z)/t| < \infty \qquad if\ p \le l$

or

$|(y+z)/t| < 1 \qquad if\ p = l + 1,$

be such that $\mathrm{R}(\sigma) > 0$ *and* $\mathrm{R}(\eta) > \Re(\rho) > -1$, $\alpha_i \neq 0, -1, -2, \ldots; i = 1, 2, \ldots, l$. *then the following fractional integral formula holds true:*

$$\left(K_{x,\infty}^{\sigma,\eta} t^{\rho-1} {}_pF_l\left[\begin{array}{c} a_1, \ldots a_p \\ \alpha_1, \ldots, \alpha_l \end{array}; \frac{y+z}{t}\right]\right)(x)$$
$$= x^{\rho-\nu-1} \frac{\Gamma(1-\rho+\eta)}{\Gamma(1-\rho+\eta+\sigma)} \times {}_{p+1}F_{l+1}\left[\begin{array}{c} a_1, \ldots a_p, 1-\rho+\eta; \\ \alpha_1, \ldots, \alpha_l, 1-\rho+\eta+\sigma; \end{array} \frac{y+z}{x}\right] \tag{2.19}$$

Further, for $\nu = -\sigma$ in (2.16) and (2.17) then these Saigo fractional integrals reduce to the following Riemann-Liouville and the Weyl type fractional integral operators as given in following results:

Corollary 13. *Let* $x > 0, \sigma, \rho \in \mathrm{C}$, $p, l \in \mathrm{N}_0$ *and*

$|t(y+z)| < \infty \qquad if\ p \le l$

or

$|t(y+z)| < 1 \qquad if\ p = l + 1,$

be such that $\mathrm{R}(\sigma) > 0$ *and* $\mathrm{R}(\rho + r) > 0$, $\alpha_i \neq 0, -1, -2, \ldots; i = 1, 2, \ldots, l$. *then the following fractional integral formula holds true:*

$$\left(R_{0,x}^{\sigma} t^{\rho-1} {}_pF_l\left[\begin{array}{c} a_1, \ldots a_p \\ \alpha_1, \ldots, \alpha_l \end{array}; t(y+z)\right]\right)(x)$$
$$= x^{\rho+\sigma-1} \frac{\Gamma(\rho)}{\Gamma(\rho+\sigma)} \times {}_{p+1}F_{l+1}\left[\begin{array}{c} a_1, \ldots a_p, \rho; \\ \alpha_1, \ldots, \alpha_l, \rho+\sigma; \end{array} x(y+z)\right] \tag{2.20}$$

Corollary 14. *Let* $x > 0, \sigma, \rho \in \mathrm{C}$, $p, l \in \mathrm{N}_0$ *and*

$|(y+z)/t| < \infty \qquad if\ p \le l$

or

$|(y+z)/t| < 1 \qquad if\ p = l + 1,$

be such that $\Re(\rho + r) > \Re(\alpha) > -1$, $\alpha_i \neq 0, -1, -2, \ldots; i = 1, 2, \ldots, l$. *then the following fractional integral formula holds true:*

$$\left(W_{x,\infty}^{\sigma} t^{\rho-1} {}_pF_l\left[\begin{array}{c} a_1, \ldots a_p \\ \alpha_1, \ldots, \alpha_l \end{array}; \frac{y+z}{t}\right]\right)(x)$$
$$= x^{\rho-\nu-1} \frac{\Gamma(1-\rho+\sigma)}{\Gamma(1-\rho)} \times {}_{p+1}F_{l+1}\left[\begin{array}{c} a_1, \ldots a_p, 1-\rho+\sigma; \\ \alpha_1, \ldots, \alpha_l, 1-\rho; \end{array} \frac{y+z}{x}\right] \tag{2.21}$$

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